Set-Up Slide

(Do not read.)

Possible World Semantics for First Order LP

Melvin Fitting

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In propositional Justification Logics necessity is split into a family of complex terms called *justifications*.

Instead of $\Box A$ one finds $t{:}A$, which can be read “ t is a justification for A“

The structure of t embodies how we come to know A, or to verify A .

Many standard modal logics
have justification counterparts.

The modal/justification connection
is via *realization* theorems.

Modal (implicit) operators
can be replaced by
justification (explicit) terms,
turning modal theorems
into justification logic theorems.

We won’t go into this now.

But it is a main source of interest in justification logics.

Modal operators conceal explicit reasons, and these can be uncovered.

Recently (2011)
Artemov and Yavorskaya
added quantification to this.

In this talk we give a
Kripke-style semantics
for their logic.

Two kinds of variables

In a first-order proof,
free variables play two different roles.

One is that of a formal symbol.

Universal Generalization lets us conclude $(\forall x)A(x)$ from a proof of $A(x)$.

Here x is a syntactic object with no inherent meaning.

The second role is a place-holder.

We can turn a proof of $A(x)$ into a proof of $A(3)$ by substituting 3 for free x occurrences.

(Provided universal generalization on x wasn't used.)

Formal symbol or
place-holder in a template.

The two are different,
incompatible even.

The Artemov-Yavorskaya
axiomatics separates the roles,
and so does the present semantics.

In propositional *LP* (logic of proofs),
if t is a justification term
and A is a formula,
then $t{:}A$ is a formula.

Read it as t justifies A.

In first-order, *FOLP*,
we have $t{:}_X A$,
where X is a finite set of variables.

Now t justifies
(is a proof of) A in which
variables in X can be substituted for
(they are place-holders)
and so can't be quantified
(variables not in X can be).

Syntax

A countable set of predicate symbols,
of any arity.

No function symbols, constants, or equality.

Countably many variables, typically $x_1, x_2, x_3, \ldots$.

Justification terms (proof terms) are built up:

Proof variables, $p_1, p_2, p_3, \ldots$,

Complex terms, $t \cdot s, t + s, !t$,

Proof constants, $c_1, c_2, c_3, \ldots$,

Proof variables 'stand for' arbitrary justifications,
proof constants justify axioms,
$t \cdot s$ corresponds to modus ponens applications,
$t + s$ is a 'weakening' operation, and
$!t$ justifies that t justifies what it is supposed to.

And for first-order,
for each individual variable x,
there is a function symbol gen_x on proof terms.

Note: x does not occur in gen_x.

If t justifies A,
then $\mathsf{gen}_x(t)$ justifies $(\forall x)A$.

If t is a justification term,
X is a finite set of individual variables,
and A is a formula,
then $t{:}_X A$ is a formula.

Free individual variable occurrences
in $t{:}_X A$ are the
free occurrences in A,
provided the variables also occur in X,
together with the occurrences in X itself.

For substitution purposes,
an individual variable y is
free for x in $t{:}_X A$ provided
y is free for x in A,
and if y occurs free in A,
then $y \in X$.

Axioms

A1 classical axioms of first order logic

A2 $t{:}_{Xy}A \supset t{:}_X A$, provided y does not occur free in A

A3 $t{:}_X A \supset t{:}_{Xy}A$

B1 $t{:}_X A \supset A$

B2 $s{:}_X(A \supset B) \supset (t{:}_X A \supset (s \cdot t){:}_X B)$

B3 $t{:}_X A \supset (t+s){:}_X A$, $s{:}_X A \supset (t+s){:}_X A$

B4 $t{:}_X A \supset !t{:}_X t{:}_X A$

B5 $t{:}_X A \supset \mathsf{gen}_x(t){:}_X \forall x A$, provided $x \notin X$

Rules of Inference

R1 $\vdash A, A \supset B \ \Rightarrow \ \vdash B$

R2 $\vdash A \ \Rightarrow \ \vdash \forall x A$

R3 $\vdash c{:}_{\emptyset} A$, where A is an axiom and c is a proof constant

Constant Specifications

A *constant specification* is a set $\mathcal{C}$ of formulas $c{:}_{\emptyset} A$, where A is an axiom.

A proof *meets* constant specification $\mathcal{C}$ if all applications of rule $\mathbf{R3}$ introduce members of $\mathcal{C}$.

Semantics

For FOLP we use
standard first-order
monotonic
Kripke semantics.

Basic Kripke Semantic Ideas

Propositional connectives are truth functional at each world, as usual.

What about justification terms?

Consider $t{:}_{\{x,y\}}Q(x, y, z, w)$ at at possible world Γ, as an example.

In $t{:}_{\{x,y\}}Q(x, y, z, w)$ the occurrences of x and y are free, but not those of z or w.

We'll allow members of model domains to appear in formulas, like constants, instead of using valuation machinery.

So, what will it mean for $t{:}_{\{a,b\}}Q(a, b, z, w)$ to be true at Γ, where a, b are in the domain of Γ?

There are two conditions,
one syntactic, one semantic.

Why syntactic?

Modal semantics works with propositions, not formulas.
Equivalent formulas evaluate the same at each world.
But different formulas might have distinct justifications.

A proof of $X \wedge Y$ is different than
a proof of $\neg(\neg X \vee \neg Y)$.

Evidence Functions

From propositional justification logics,
we bring in *evidence functions* $\mathcal{E}$.

Propositionally, $\mathcal{E}(t, A)$ is
the set of possible worlds
at which t serves as
meaningful evidence for A.

Meaningful evidence is not conclusive,
merely pertinent (informally speaking).

Kripke/semantic

Propositionally we take $t{:}A$ to be
true at a possible world if
A is true at all accessible worlds,
and t is relevant evidence for A at that world.

syntactic

With first-order machinery present,
this becomes more complicated,
because of the two roles variables
can play in proofs.

In $t{:}_{\{x,y\}}Q(x,y,z,w)$ the variables x, y can be substituted for, but z, w are those to which universal generalization applies.

For the first part, we will only talk about substitution instances which replace x, y.

We consider $t{:}_{\{a,b\}}Q(a,b,z,w)$ at Γ, where a, b are in the domain of Γ.

For the second part,
for $t{:}_{\{a,b\}}Q(a,b,z,w)$,
to be true at Γ we require,
at every Δ accessible from Γ,
for every c,d in the domain of Δ,
$Q(a,b,c,d)$ must be true at Δ.

No matter what work we do
(the move from Γ to Δ),
no matter what mathematical objects we construct
(c and d)
$Q(a,b,x,y)$ will hold for these instances.

To summarize:

$t{:}_{\{a,b\}}Q(a,b,z,w)$ is true at Γ provided

1. t is meaningful evidence for $Q(a,b,z,w)$ at Γ, that is, $\Gamma \in \mathcal{E}(t, Q(a,b,z,w))$
2. for every Δ accessible from Γ, and for every c, d in the domain of Δ, $Q(a,b,c,d)$ is true at Δ.

Now the formal details.

Skeletons

$$\langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$$

$\mathcal{G}$ possible worlds

$\mathcal{R}$ reflexive, transitive accessibility relation

$\mathcal{D}$ domain function

$$\mathcal{D} : \mathcal{G} \rightarrow \text{ non-empty sets}$$

We require monotonicity:

$\Gamma \mathcal{R} \Delta$ implies $\mathcal{D}(\Gamma) \subseteq \mathcal{D}(\Delta)$

Domain of skeleton:

$\mathcal{D}^* = \cup\{\mathcal{D}(\Gamma) \mid \Gamma \in \mathcal{G}\}$

Constants (from a model domain)

We will allow members of a skeleton domain to appear as constants in formulas.

This avoids the use of valuation functions, and makes things generally more intuitive.

Models

Let $\mathcal{S} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ be a skeleton. A *model* based on it is $\mathcal{M} = \langle \mathcal{S}, \mathcal{I}, \mathcal{E} \rangle$ where:

semantic

$\mathcal{I}$ is an *interpretation function*, assigning a relation on $\mathcal{D}(\Gamma)$ to each relation symbol, at each possible world Γ.

syntactic

$\mathcal{E}$ is an *evidence function*, assigning to each justification term t, and to each formula A (with model constants) some set $\mathcal{E}(t, A)$ of possible worlds.

Special Condition:

if $\Gamma \in \mathcal{E}(t, A)$ then
all model constants in A are
from $\mathcal{D}(\Gamma)$.

The Idea:

If $\Gamma \in \mathcal{E}(t, A)$ think of t as *relevant* evidence for A at Γ.

Not conclusive, just relevant.

Let's say a formua A,
with model constants,
lives in Γ if
all constants in A are
from $\mathcal{D}(\Gamma)$.

Evidence Function Conditions

Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E} \rangle$ be an FOLP model. We require the evidence function to meet the following conditions.

$\cdot$ **Condition** $\mathcal{E}(s, A \supset B) \cap \mathcal{E}(t, A) \subseteq \mathcal{E}((s \cdot t), B)$.

$+$ **Condition** $\mathcal{E}(s, A) \cup \mathcal{E}(t, A) \subseteq \mathcal{E}((s + t), A)$.

$\mathcal{R}$ **Closure Condition** $\Gamma \mathcal{R} \Delta$ and $\Gamma \in \mathcal{E}(t, A)$ imply $\Delta \in \mathcal{E}(t, A)$.

! **Condition** $\mathcal{E}(t, A) \subseteq \mathcal{E}(!t, t{:}_X A)$ where X is the set of all members of $\mathcal{D}^*$ that appear in A.

Instantiation Condition If $a \in \mathcal{D}(\Gamma)$ and $\Gamma \in \mathcal{E}(t, A(x))$ then $\Gamma \in \mathcal{E}(t, A(a))$.

$\mathbf{gen}_x$ **Condition** $\mathcal{E}(t, A) \subseteq \mathcal{E}(\mathsf{gen}_x(t), \forall x A)$.

An evidence function
(and a model)
meets a constant specification
provided:

$c{:}_{\emptyset} A \in \mathcal{C}$ implies $\Gamma \in \mathcal{E}(c, A)$
whenever A lives in Γ

Truth (at a world)

$$\mathcal{M}, \Gamma \Vdash A$$

model

possible world

formula (allowing model constants)

A is true at Γ in $\mathcal{M}$

Assume formulas are closed (but allow constants from the model)

- (atomic) $\mathcal{M}, \Gamma \Vdash Q(a,b) \iff \langle a,b \rangle \in \mathcal{I}(\Gamma, Q)$
- $\mathcal{M}, \Gamma \nVdash \bot$
- propositional connectives are classical, at each world
- $\mathcal{M}, \Gamma \Vdash \forall x A(x) \iff \mathcal{M}, \Gamma \Vdash A(a)$ for every $a \in \mathcal{D}(\Gamma)$

- $\mathcal{M}, \Gamma \Vdash t{:}_X A(x,y) \iff$

  1. $\Gamma \in \mathcal{E}(t, A(x,y))$ and
  2. $\mathcal{M}, \Delta \Vdash A(a,b)$ for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$ and for every a, b in $\mathcal{D}(\Delta)$.

Validity

Let A be a closed formula with no domain constants.

A is *valid* in model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}, \mathcal{E} \rangle$ if $\mathcal{M}, \Gamma \Vdash A$ for all $\Gamma \in \mathcal{G}$.

A formula with free individual variables is valid if its universal closure is.

Example 1

$t{:}_{Xy}A \supset t{:}_X A$ is an axiom,
provided y does not occur free in A.

Note:
not free.

The proviso is necessary.

We show non-validity of
$(\forall x)(\forall y)[t{:}_{\{x,y\}}Q(x,y) \supset t{:}_{\{x\}}Q(x,y)]$.

We do this by showing failure of an instance.

Call an evidence function $\mathcal{E}$ *universal* provided $\Gamma \in \mathcal{E}(t, A)$ whenever A lives in Γ.

This is, in fact, an evidence function.

Here's a model, schematically.

Reflexivity is not shown, and the evidence function is universal.

$$
\begin{tikzcd}
\Gamma \arrow[d] & \boxed{a, b} \Vdash Q(a,b) \\
\Delta & \boxed{a, b, c} \Vdash Q(a,b)
\end{tikzcd}
$$

$$\mathcal{M}, \Gamma \nVdash t{:}_{\{a,b\}} Q(a,b) \supset t{:}_{\{a\}} Q(a,y)$$

because

$$\mathcal{M}, \Gamma \Vdash t{:}_{\{a,b\}} Q(a,b)$$

because $\Gamma \in \mathcal{E}(t, Q(a,b))$

and $\mathcal{M}, \Gamma \Vdash Q(a,b)$ and $\mathcal{M}, \Delta \Vdash Q(a,b)$.

$$\mathcal{M}, \Gamma \nVdash t{:}_{\{a\}} Q(a,y)$$

because we do not have $\mathcal{M}, \Delta \Vdash Q(a,c)$

Example 2

$t{:}_X A \supset \mathsf{gen}_x t{:}_X \forall x A$
is an axiom, where $x \notin X$

The proviso is necessary.

$(\forall x)[t{:}_{\{x\}} Q(x) \supset \mathsf{gen}_x t{:}_{\{x\}} (\forall x) Q(x)]$
is not valid.

$$\Gamma \quad \boxed{a, b} \Vdash Q(a)$$

Reflexivity not shown,
and evidence function is universal.

$$\mathcal{M}, \Gamma \Vdash t{:}_{\{a\}} Q(a)$$
because $\Gamma \in \mathcal{E}(t, Q(a))$
and $\mathcal{M}, \Gamma \Vdash Q(a)$

$$\mathcal{M}, \Gamma \nVdash \mathsf{gen}_x t{:}_{\{a\}} \forall x Q(x)$$
because otherwise $\mathcal{M}, \Gamma \Vdash \forall x Q(x)$ (reflexivity)
but $\mathcal{M}, \Gamma \nVdash Q(b)$

Soundness

We show validity of two representative axioms.

In what follows,
assume $\mathcal{M}$ is an FOLP model.

We work with
special cases
that are sufficiently general.

Axiom A3

$t{:}_X A \supset t{:}_{X_y} A$ is valid.

Special case:
$t{:}_{\{x\}} A(x, y, z) \supset t{:}_{\{x,y\}} A(x, y, z)$.

Instance of special case, to show:
$\mathcal{M}, \Gamma \Vdash t{:}_{\{a\}} A(a, y, z) \supset t{:}_{\{a,b\}} A(a, b, z)$.

Note: $a, b \in \mathcal{D}(\Gamma)$.

Suppose $\mathcal{M}, \Gamma \Vdash t{:}_{\{a\}} A(a, y, z)$.

Then $\Gamma \in \mathcal{E}(t, A(a, y, z))$.

$\Gamma \in \mathcal{E}(t, A(a, b, z))$ (Instantiation Condition)

And for every accessible Δ,
$\mathcal{M}, \Delta \Vdash A(a, c, d)$
for every $c, d \in \mathcal{D}(\Delta)$.

$b \in \mathcal{D}(\Delta)$ (Monotonicity)

So $\mathcal{M}, \Delta \Vdash A(a, b, d)$
for every $d \in \mathcal{D}(\Delta)$.

Then $\mathcal{M}, \Gamma \Vdash t{:}_{\{a,b\}} A(a, b, z)$.

Axiom B5

$t{:}_X A \supset \mathsf{gen}_x(t){:}_X \forall x A$ is valid
provided $x \notin X$

Special case:
$t{:}_{\{y\}} A(x, y, z) \supset \mathsf{gen}_x(t){:}_{\{y\}} \forall x A(x, y, z)$

Instance that we show:
$\mathcal{M}, \Gamma \Vdash t{:}_{\{b\}} A(x, b, z) \supset \mathsf{gen}_x(t){:}_{\{b\}} \forall x A(x, b, z)$

where $b \in \mathcal{D}(\Gamma)$.

Assume

$\mathcal{M}, \Gamma \Vdash t{:}_{\{b\}} A(x, b, z)$

Then $\Gamma \in \mathcal{E}(t, A(x, b, z))$

$\Gamma \in \mathcal{E}(\mathsf{gen}_x(t), \forall x A(x, b, z))$ (gen_x condition)

and for every accessible Δ,

$\mathcal{M}, \Delta \Vdash A(a, b, c)$

for all $a, c \in \mathcal{D}(\Delta)$.

$\mathcal{M}, \Delta \Vdash \forall x A(x, b, c)$

for all $c \in \mathcal{D}(\Delta)$

Then $\mathcal{M}, \Gamma \Vdash \mathsf{gen}_x t{:}_{\{b\}} \forall x A(x, b, z)$,

Completeness

Actually, we won't prove completeness here.
It is too long.

But here are some comments.

Soundness is with respect to any constant specification.
Completeness requires something more restricted.

A constant specification $\mathcal{C}$ is *axiomatically appropriate* if, for every axiom A there is a proof constant c such that $c{:}_{\emptyset}A \in \mathcal{C}$.

(Very common requirement)

An *internalization theorem* holds for FOLP provided an axiomatically appropriate constant specification is assumed.

Two formulas are *variable variants* if
each can be turned into the other
by a renaming of free and bound individual variables.

A constant specification $\mathcal{C}$ is
variant closed if
whenever A and B are variable variants,
$c{:}_{\emptyset}A \in \mathcal{C}$ if and only if $c{:}_{\emptyset}B \in \mathcal{C}$.

Our completeness proof uses a Henkin construction.

We extend the basic language with ‘witnesses.’

A variant closed constant specification naturally extends to a language with added witnesses.

Our completeness proof needs
a constant specification that is
axiomatically appropriate
and
variant closed.

All details omitted in this talk,
but there is a paper.

Fully Explanatory

Let A be a formula
with no free individual variables,
but with constants from the model.

Suppose that A lives in Γ and
$\mathcal{M}, \Delta \Vdash A$ for every $\Delta \in \mathcal{G}$.
(So A is *necessary* at Γ.)

Can't be said in the language itself.

We might say
t *justifies this necessity* if
$\mathcal{M}, \Gamma \Vdash t{:}_X A$,
where X is the set of domain constants appearing in A.

Call a model

fully explanatory

if every necessity is justified.

Completeness holds

with respect to

fully explanatory models

(just as it does propositionally).

Mkrtychev Models

These are essentially
one-world models.

Completeness with respect
to Mkrtychev Models holds,
again as in the propositional case.

Conclusion

The same set of semantic tools available propositionally is also available for first-order.

Propositionally, LP is one of a *family* of justification logics, and the same is true for first-order.

Propositional justification logics J, JT, J4, and some others, have monotonic first-order versions—all the results given here carry over to them.

The situation with quantified logics involving symmetry is open.

Their semantics would be constant domain.

Work is underway on first-order justification logics with constant domain semantics,

but this is very much still in progress.

Paper Available

A paper

Possible World Semantics for First Order LP

is available on my web site.

Thank You